Utility Functions
Part IV - The Quadratic Utility Function

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In this white paper we will define the exponential utility function. To that end we will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

We are given the following investment payoffs and market assumptions...

Table 1: Investment Payoffs

Symbol	Payoff	Probability
W_1	200	0.40
W_2	500	0.30
W_3	700	0.20
W_4	950	0.10

Table 2: Market Assumptions

Symbol	Description	Value
μ	Annual risk-free interest rate (%)	4.50
κ	Annual expected investment return (%)	8.00
T	Investment term in years (#)	3.00

Questions:
1. Given a discount rate of 8.00%, what is the value of the certainty equivalent at time T and at time zero?
2. What is the value of the scalar alpha?
3. Graph utility, marginal utility, and risk aversion.

Expected Wealth

We will define the variable p_i to be the probability of realizing wealth W_i (the i'th random investment payoff). The equations for expected wealth and expected wealth squared is...

$$\mathbb{E}\Big[W\Big] = \sum_{i=1}^{n} p_i\, W_i \text{ ...and... } \mathbb{E}\Big[W^2\Big] = \sum_{i=1}^{n} p_i\, W_i^2 \tag{1}$$

Using Equation (1) above and the data in Table 1 above, the expected increase in wealth from our investment as defined in Table 1 above is...

$$\mathbb{E}\Big[W\Big] = 200 \times 0.40 + 500 \times 0.30 + 700 \times 0.20 + 950 \times 0.10 = 465.00 \tag{2}$$

Quadratic Utility Function

We will define the variable α to be a scalar whose value is greater than zero. The equation for our quadratic untility function and it's first and second derivatives with respect to the random increase in wealth (W_i) are the following equations...

$$U(W_i) = W_i - \alpha\, W_i^2 \text{ ...where... } U'(W_i) = 1 - 2\,\alpha\, W_i \text{ ...and... } U''(W_i) = -2\,\alpha \tag{3}$$

Using Equation (3) above, we can make the following statements...

$$U(W_i) = 0 \text{ ...when... } W_i = 0 \text{ ...and... } U(W_i) > 0 \text{ ...when... } W_i > 0 \tag{4}$$

For investors who prefer more wealth to less wealth, the quadratic utility function may only represent investor preferences over a restricted range of wealth. To be consistent with nonsatiation, restrictions must be placed on the value of the scalar α. Using Equation (3) above, the equation for our initial guess value of scalar α where W_{max} is our maximum payoff and b is the marginal utility of our maxiumum payoff is...

$$\text{if... } U'(W_{max}) = b \text{ ...then... } \alpha = \frac{1}{2}\,(1-b)\,W_{max}^{-1} \text{ ...where... } 1 > b \geq 0 \tag{5}$$

We will define the variable λ to be the Arrow-Pratt measure of risk aversion. Using Equation (3) above, the equation for this measure of risk aversion is... [1]

$$\lambda = -\frac{U''(W_i)}{U'(W_i)} = \frac{2\,\alpha}{1-2\,\alpha\,W_i} \text{ ...where... } \frac{\delta\lambda}{\delta W_i} = 4\,\alpha\,(1-2\,\alpha\,W_i)^{-1} = \frac{4\,\alpha^2}{\text{marginal utility squared}} \tag{6}$$

Using Equations (5) and (6) above, the derivative of λ is always positive, which implies increasing relative risk aversion. Therefore, investors will decrease the percentage of wealth allocated to risky assets as they get wealthier. Thus, quadratic utility functions have characteristics that are undesirable.

Expected Utility

Using Equations (1) and (3) above, the equation for the expected utility of wealth is... [1]

$$\mathbb{E}\Big[U(W)\Big] = \sum_{i=1}^{n} p_i\,U(W_i) = \sum_{i=1}^{n} p_i \Big(W_i - \alpha\,W_i^2\Big) = \mathbb{E}\Big[W\Big] - \alpha\,\mathbb{E}\Big[W^2\Big] \tag{7}$$

Using Equation (7) above, the equation for the derivative of the expected utility of wealth with respect to the scalar parameter α is...

$$\frac{\delta}{\delta\,\alpha}\,\mathbb{E}\Big[U(W)\Big] = -\mathbb{E}\Big[W^2\Big] \tag{8}$$

Certainty Equivalent

We will define the variable CE to be the value of the certainty equivalent at time T. The dollar value of the certainty equivalent is such that the utility of the certainty equivalent is equal to the utility of expected wealth. This statement in equation form is... [1]

$$U(CE) = \mathbb{E}\Big[U(W)\Big] \tag{9}$$

Using Equations (3) and (7) above, we can rewrite Equation (9) above as...

$$CE - \alpha\,CE^2 = \mathbb{E}\Big[W\Big] - \alpha\,\mathbb{E}\Big[W^2\Big] \tag{10}$$

In Table 2 above we defined the variable μ to be the risk-free rate and the variable κ to be the risk-adjusted discount rate. If we are given the discount rate κ then using Equation (1) above the equation for the value of the certainty equivalent at time T is...

$$\text{if... } \kappa = \left(\mathbb{E}\Big[W\Big] \Big/ CE\,(1+\mu)^{-T}\right)^{1/T} - 1 \text{ ...then... } CE = \mathbb{E}\Big[W\Big]\left(\frac{1+\mu}{1+\kappa}\right)^{T} \tag{11}$$

Given the value of the certainty equivalent at time T (Equation (11) above) we want to solve Equation (10) above for the value of the scalar α. We will make the following scalar parameter definitions...

$$\alpha = \text{Actual value of the scalar alpha ...and... } \hat{\alpha} = \text{Guess value of the scalar alpha} \tag{12}$$

Using Equations (10) and (12) above, we will define the function $f(\alpha)$ to be...

$$f(\alpha) = \mathbb{E}\Big[W\Big] - \alpha\,\mathbb{E}\Big[W^2\Big] - CE + \alpha\,CE^2 = 0 \tag{13}$$

Using Equations (10) and (12) above, we will define the function $f(\hat{\alpha})$ to be...

$$f(\hat{\alpha}) = \mathbb{E}\Big[W\Big] - \hat{\alpha}\,\mathbb{E}\Big[W^2\Big] - CE + \hat{\alpha}\,CE^2 \tag{14}$$

Using Equation (8) above, the derivative of Equation (14) above with respect to the scalar α is...

$$f'(\hat{\alpha}) = CE^2 - \mathbb{E}\Big[W^2\Big] \tag{15}$$

To solve for κ, the Newton-Raphson equation that we will iterate is... [2]

$$\alpha + \hat{\epsilon} = \hat{\alpha} + \frac{f(\alpha) - f(\hat{\alpha})}{f'(\hat{\alpha})} \;\Bigg|\; f(\alpha) = \text{Equation (13)},\ f(\hat{\alpha}) = \text{Equation (14)},\ f'(\hat{\alpha}) = \text{Equation (15)} \tag{16}$$

Answers To Our Hypothetical Problem

1. Given a discount rate of 8.00%, what is the value of the certainty equivalent at time T and at time zero?

Using Equations (2) and (11) above and the data in Table 1 above, the value of the certainty equivalent at time T is...

$$CE = 465.00 \times \left(\frac{1 + 0.0450}{1 + 0.8000}\right)^3 = 421.24 \tag{17}$$

Using Equation (18) above and the data in Table 1 above, the value of the certainty equivalent at time zero is...

$$PVCE = 421.24 \times (1 + 0.0450)^{-3} = 369.13 \tag{18}$$

2. What is the value of the scalar alpha?

Using Equation (5) above and the data in Table 1 above, the guess value of our scalar is...

$$\alpha = \frac{1}{2} \times (1 - 0.25) \times 950.00^{-1} = 0.000395 \tag{19}$$

Using Equations (16) and (17) above and the data in Table 1 above, the value of our scalar is...

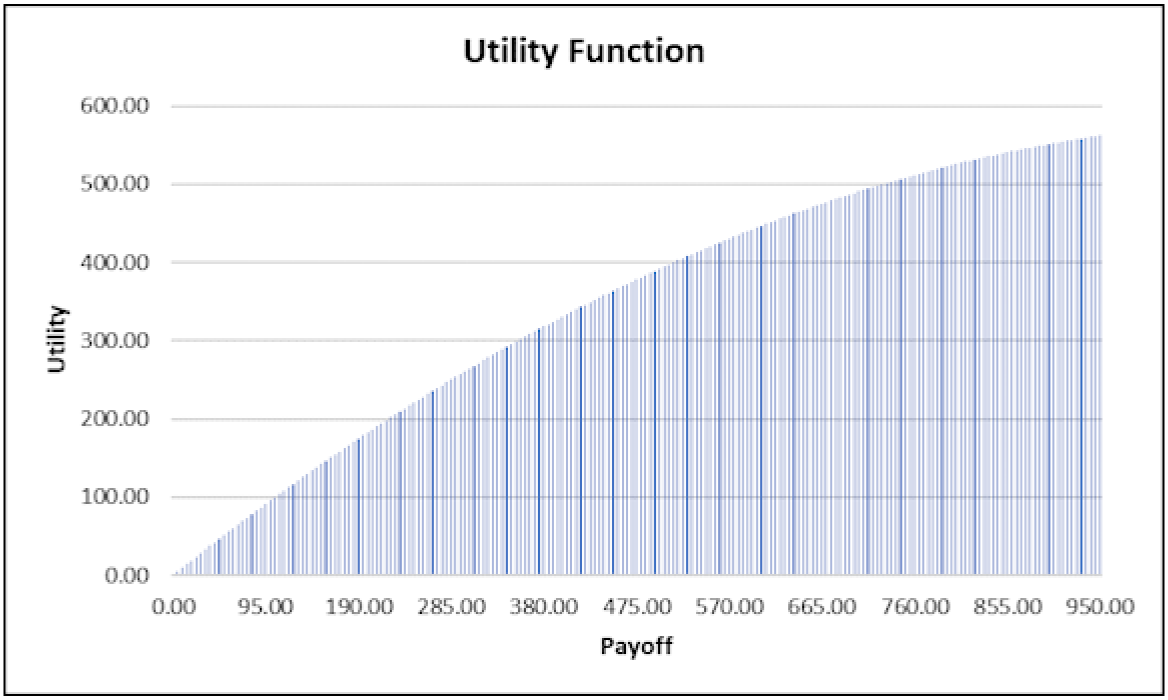
Iteration	Scalar	$f(\alpha)$	$f(\hat{\alpha})$	$f'(\hat{\alpha})$
0	0.000395	0.0000	3.5725	-101,806.08
1	0.000430	0.0000	0.0000	-101,806.08
2	0.000430	0.0000	0.0000	-101,806.08
3	0.000430	0.0000	0.0000	-101,806.08

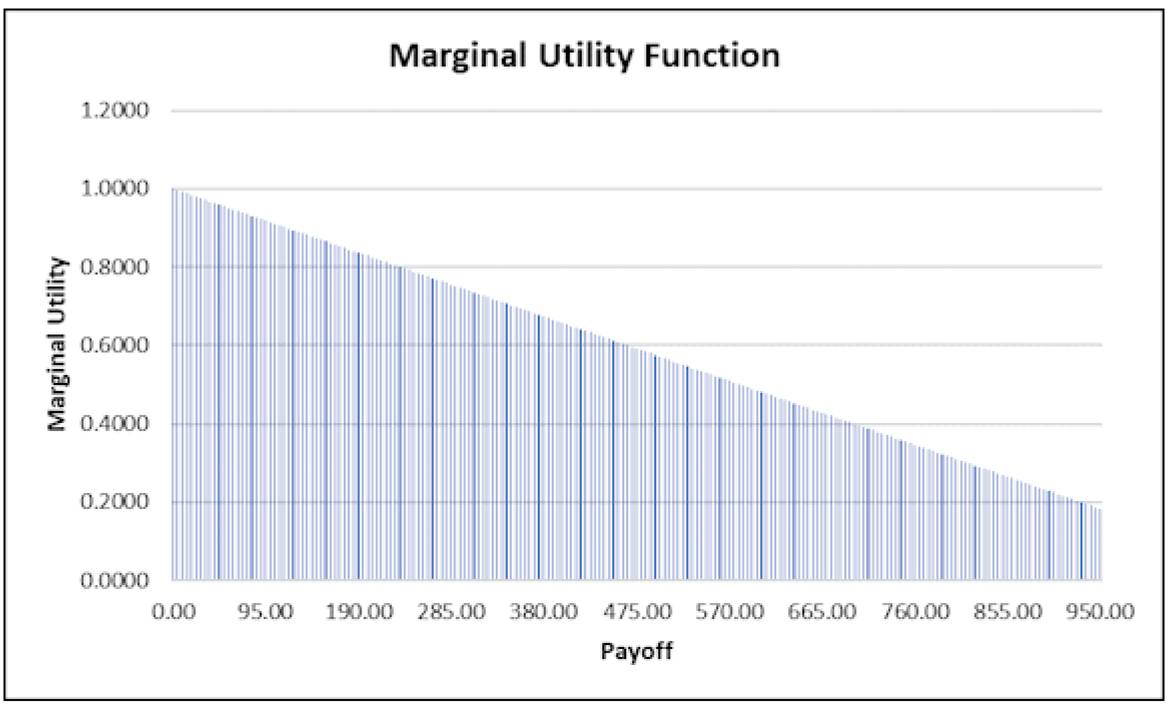
Using the results in the table above, the answer to the question is...

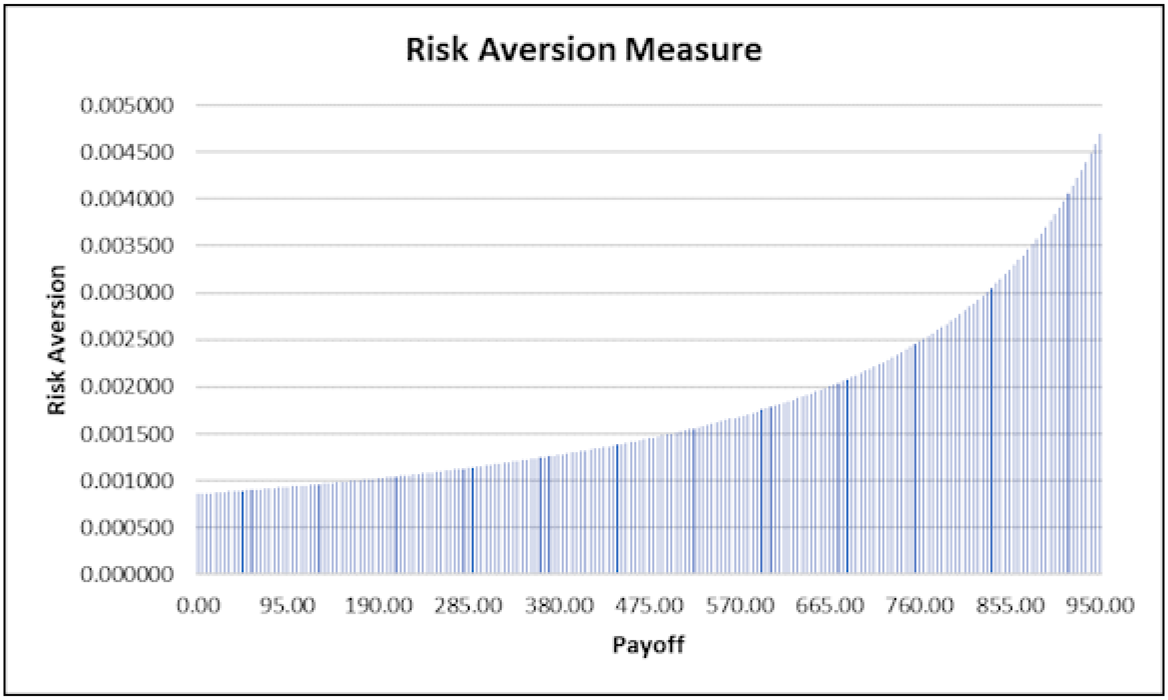
$$\text{if...}\ \ \alpha = \text{scalar value} = 0.000430\ \ \text{...then...}\ \ \kappa = \text{target discount rate} = 8.00\% \tag{20}$$

3. Graph utility, marginal utility, and risk aversion.

Using Equation (3) above and our estimate of the scalar parameter in Equation (20) above, our graphs are...







References

[1] Gary Schurman, *Introduction To Utility Funtions,* October, 2023.

[2] Gary Schurman, *The Newton-Raphson Method For Solving Non-Linear Equations,* October, 2009.